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Three-Dimensional Axisymmetric Stress Analysis of Superconducting Magnets Using Green's Function Solution

A closed-form Green's function solution for the axisymmetric stresses in an elastic coil of superconducting magnets is presented, which provides the components of stress throughout the coil and includes the shear stress in addition to the normal stresses. The Green's function method permits the development of a solution irrespective of the type of magnetic body forces within the coil. Green's functions are derived by using finite Hankel transforms appropriate for a cylindrical coil. [DOI: 10.1115/1.1345700]

1 Introduction

The prediction of stress and strain is essential for both mechanical and electrical design of high-field solenoid magnets. These magnets are designed in a variety of configurations. A superconducting magnet is one example of such magnets, which can be treated as a combination of several solenoid coils, where each coil may be reinforced by a nonconducting layer. Depending upon the geometrical specifications of a coil, magnetic fields may behave differently. These fields result in magnetic body forces, and thus stresses. Traditionally only the tangential component of the stress at the plane perpendicular to the middle section of the longitudinal axis of a coil (midplane) has been considered for design and failure analysis. The value of shear stress has been determined to be small in the midplane but it becomes larger toward the ends of the coil. In the analytical solutions available in the literature, the stress analysis has been performed for just the midplane and shear stress is assumed to be negligible ([1–4]). As a result, a three-dimensional closed-form solution is desired to understand the distribution of stresses (including shear) throughout a solenoid coil. In the present work, a general closed-form solution, using the Green's function method, is derived for an elastic, isotropic coil of a high-field solenoid magnet. This solution is applied to the important case of a superconducting magnet. This analysis is not limited to the midplane and can be used for any type of solenoid magnet.

The use of a Green's function solution is not limited to magnetic body forces. It can be applied to other axisymmetric elasticity problems for finite bodies. The Green's function solution can be used for inclusion problem in composite materials where eigenstrains may be considered as body forces ([5,6]). The Green's function solution can also be applied to specific problems in fracture mechanics and composite materials ([7–9]). Fictitious body forces can be introduced in composite materials, where the difference between the thermal expansion coefficients of the fiber and the matrix results in residual stresses.

A limited number of publications on the approximation of the three-dimensional problem are available in the literature. Some solutions are obtained by neglecting shear throughout the coil ([10]). Other solutions are based upon numerical techniques ([11]) or power series expansion of the fields and displacements ([12]). The direct analytical solutions for infinite or semi-infinite domains are not appropriate for a finite domain such as a magnet ([13,14]).

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2 Fundamental Equations for the Stress Functions

Consider an elastic isotropic coil with inside radius of a, outside radius of b, and length of $2L$ as shown in Fig. 1. Using the equilibrium equations, constitutive equations, and strain-displacement relationships, the governing equations for displacement vector $\mathbf{u}(r,z)$, for an axisymmetric distribution of body forces $\mathbf{X}(r,z)$, may be written as

$$(\lambda+\mu)\nabla(\nabla.\mathbf{u})+\mu\nabla^2\mathbf{u}+\mathbf{X}=0 \quad (1)$$

where ∇^2 is the three-dimensional Laplacian and λ and μ are (Lamé's) elastic coefficients ([15]).

From the Helmholtz theorem, any vector satisfying Eq. (1) may be resolved into a sum of a gradient and a curl

$$\mathbf{u}=\nabla\phi+\nabla\times\mathbf{A} \quad (2)$$

where $\phi(r,z)$ is a scalar potential and $\mathbf{A}(r,z)$ is a vector potential such that $\nabla.\mathbf{A}=0$. Incorporating the displacement vector from Eq. (2) into Eq. (1) yields an equation in terms of potential functions ϕ and $\mathbf{A}$.

$$(\lambda+2\mu)\nabla(\nabla^2\phi)+\mu\nabla\times(\nabla^2\mathbf{A})+\mathbf{X}=0. \quad (3)$$

The independent potential functions ϕ and $\mathbf{A}$ may be written as

$$\mathbf{A}=\alpha\nabla\times\mathbf{\Psi} \qquad \phi=\beta\nabla.\mathbf{\Psi} \quad (4)$$

where α and β are arbitrary constants, and components of the vector $\mathbf{\Psi}$ are the stress functions. Introducing Eq. (4) into Eq. (3) leads to a partial differential equation for vector $\mathbf{\Psi}$.

$$\beta(\lambda+2\mu)\nabla^4\mathbf{\Psi}+[\beta(\lambda+2\mu)+\mu\alpha]\nabla\times[\nabla\times(\nabla^2\mathbf{\Psi})]+\mathbf{X}=0 \quad (5)$$

In order to simplify Eq. (5), we may choose the arbitrary constants α and β as $-1/\mu$ and $1/(\lambda+2\mu)$, respectively. Thus, Eq. (5) reduces to the component form.

$$\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_r-\frac{4}{r^4}\frac{\partial^2\Psi_r}{\partial\theta^2}-\frac{4}{r^2}\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Psi_\theta}{\partial\theta}+X_r=0$$

$$\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_\theta-\frac{4}{r^4}\frac{\partial^2\Psi_\theta}{\partial\theta^2}+\frac{4}{r^2}\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Psi_r}{\partial\theta}+X_\theta=0$$

$$\nabla^4\Psi_z+X_z=0 \quad (6)$$

Because the geometry and loading are axisymmetric, the partial derivative with respect to tangential direction is zero, $\partial/\partial\theta=0$.

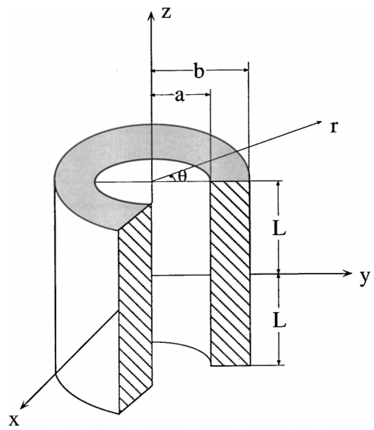

Fig. 1 Schematic diagram representing one coil of a magnet

Hence, the three partial differential equations in Eq. (6) reduce to three uncoupled partial differential equations for radial, tangential, and axial stress functions:

$$\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_r+X_r=0 \tag{7}$$

$$\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_\theta+X_\theta=0 \tag{8}$$

$$\nabla^4\Psi_z+X_z=0. \tag{9}$$

The body force in a magnet is the Lorentz force, a function of r and z related to the magnetic field, **B**, and current density, **J**, by

$$\mathbf{X}=\mathbf{J}\times\mathbf{B}. \tag{10}$$

For an axisymmetric distribution of Lorentz force, $\mathbf{J}=J_\theta\mathbf{e}_\theta$ and $\mathbf{B}=B_r\mathbf{e_r}+B_z\mathbf{e_z}$. Thus, the vector product of the **J** and **B** leads to

$$X_r=J_\theta B_z \quad X_\theta=0 \quad X_z=-J_\theta B_r. \tag{11}$$

In the absence of a tangential magnetic body force in Eq. (8), the tangential stress function will be zero, resulting in a zero tangential displacement $\mathbf{u}_\theta$.

3 Finite Hankel Transform

The partial differential equations represented by Eqs. (7) and (9) may be solved by using finite Hankel transforms ([16]). The finite Hankel transform of order n of function $f(r)$ on a closed finite interval $[a, b]$ is defined by

$$\Re_n[f(r)]=\overline{f}(\zeta_i)=\int_a^b rf(r)K_n(\zeta_i,r)dr \tag{12}$$

where ζ_i is a root of the transcendental equation

$$J_n(\zeta_i a)Y_n(\zeta_i b)-J_n(\zeta_i b)Y_n(\zeta_i a)=0 \tag{13}$$

and $K_n(\zeta_i,r)$ is the Fourier Bessel kernel.

$$K_n(\zeta_i,r)=[J_n(\zeta_i r)Y_n(\zeta_i b)-J_n(\zeta_i b)Y_n(\zeta_i r)] \tag{14}$$

The inverse transform for the finite Hankel transform is

$$f(r)=\Re_n^{-1}[\overline{f}(\zeta_i)]=\frac{\pi^2}{2}\sum_i\frac{\zeta_i^2J_n^2(\zeta_i a)}{J_n^2(\zeta_i a)-J_n^2(\zeta_i b)}\overline{f}(\zeta_i)K_n(\zeta_i,r). \tag{15}$$

where the summation is extended over all positive roots ζ_i. The finite Hankel transform of a Laplacian of $f(r)$ (in cylindrical coordinate) is given by

$$\Re_n\left[\left(\nabla^2-\frac{n^2}{r^2}\right)f\right]=\Re_n\left[\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)f\right]=\frac{2}{\pi}\left(f(b)-\frac{J_n(\zeta_i b)}{J_n(\zeta_i a)}f(a)\right)-\zeta_i^2\Re_n[f]. \tag{16}$$

4 Radial Green's Function

The Radial Green's function for Ψ_r, radial stress function, is obtained by solving Eq. (7) with Dirichlet homogeneous boundary conditions. Applying the finite Hankel transform (of order one) to each term of Eq. (7) yields

$$\Re_1\left[\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_r\right]=-\Re_1[X_r]. \tag{17}$$

Use of Eq. (16) in Eq. (17) results in

$$\left(-\zeta_{1i}^2+\frac{d^2}{dz^2}\right)^2\overline{\Psi}_r(\zeta_{1i},z)=-\overline{X}_r(\zeta_{1i},z) \tag{18}$$

where ζ_{1i} is a root of the transcendental equation

$$J_1(\zeta_{1i}a)Y_1(\zeta_{1i}b)-J_1(\zeta_{1i}b)Y_1(\zeta_{1i}a)=0 \tag{19}$$

and

$$\overline{\Psi}_r(\zeta_{1i},z)=\int_a^b r\Psi_r(r,z)K_1(\zeta_{1i},r)dr \tag{20}$$

$$\overline{X}_r(\zeta_{1i},z)=\int_a^b rX_r(r,z)K_1(\zeta_{1i},r)dr, \tag{21}$$

where

$$K_1(\zeta_{1i},r)=[J_1(\zeta_{1i}r)Y_1(\zeta_{1i}b)-J_1(\zeta_{1i}b)Y_1(\zeta_{1i}r)] \tag{22}$$

is the Fourier Bessel kernel. An additional transform (in the axial direction) is needed for solving Eq. (18). Considering the radial body force is an even function of z and the interval is finite $[-L,L]$, an appropriate transform is the finite Fourier cosine transform. By introducing the finite Fourier cosine transform (in the axial direction) of Eq. (17),

$$\Im_C\left[\left(-\zeta_{1i}^2+\frac{\partial^2}{\partial z^2}\right)^2\overline{\Psi}_r(\zeta_{1i},z)\right]=-\Im_C[\overline{X}_r(\zeta_{1i},z)],$$

the differential equation is converted into the algebraic equation

$$\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)^2\bar{\bar{\Psi}}_r(\zeta_{1i},n)=-\bar{\bar{X}}_r(\zeta_{1i},n) \tag{23}$$

where n is an integer. The functions

$$\bar{\bar{\Psi}}_r(\zeta_{1i},n)=\int_{-L}^{L}\overline{\Psi}_r(\zeta_{1i},z)\cos\frac{n\pi z}{L}dz \tag{24}$$

$$\bar{\bar{X}}_r(\zeta_{1i},n)=\int_{-L}^{L}\overline{X}_r(\zeta_{1i},z)\cos\frac{n\pi z}{L}dz \tag{25}$$

are the finite Fourier cosine transforms of $\overline{\Psi}_r(\zeta_{1i},z)$ and $\overline{X}_r(\zeta_{1i},z)$. The inverse finite Fourier cosine transform of $\bar{\bar{\Psi}}_r(\zeta_{1i},n)$ and inverse finite Hankel transform of $\overline{\Psi}_r(\zeta_{1i},z)$ are defined by

$$\overline{\Psi}_r(\zeta_{1i},z)=\Im_C^{-1}[\bar{\bar{\Psi}}_r(\zeta_{1i},n)]=\frac{1}{2L}\bar{\bar{\Psi}}_r(\zeta_{1i},0)+\frac{1}{L}\sum_{n=1}^{\infty}\bar{\bar{\Psi}}_r(\zeta_{1i},n)\cos\frac{n\pi z}{L} \tag{26}$$

$$\Psi_r(r,z)=\Re_1^{-1}[\bar{\Psi}_r(\zeta_{1i},z)]$$
$$=\frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{1i}^2J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}\bar{\Psi}_r(\zeta_{1i},z)K_1(\zeta_{1i},r). \qquad (27)$$

Substitution of Eqs. (26) and (27) into Eq. (23) yields an equation for the radial stress function:

$$\Psi_r(r,z)=-\frac{\pi^2}{2}\sum_{i=1}^{\infty}\left[\frac{\zeta_{1i}^2J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}\right]\left[\frac{1}{2L\zeta_{1i}^4}\bar{\bar{X}}_r(\zeta_{1i},0)+L^3\sum_{n=1}^{\infty}\frac{1}{(L^2\zeta_{1i}^2+n^2\pi^2)^2}\bar{\bar{X}}_r(\zeta_{1i},n)\cos\frac{n\pi z}{L}\right]K_1(\zeta_{1i},r). \qquad (28)$$

Introducing Eqs. (21) and (25) into Eq. (28) gives the solution of the radial stress function $\Psi_r(r,z)$ in terms of the radial body force $X_r(r,z)$.

$$\Psi_r(r,z)=-\frac{\pi^2}{2}\sum_{i=1}^{\infty}\left\{\frac{1}{2L\zeta_{1i}^4}\int_{-L}^{L}\int_{a}^{b}r'X_r(r',z')K_1(\zeta_{1i},r')dr'dz'+L^3\sum_{n=1}^{\infty}\left[\frac{1}{(L^2\zeta_{1i}^2+n^2\pi^2)^2}\cos\frac{n\pi z}{L}\int_{-L}^{L}\int_{a}^{b}r'X_r(r',z')K_1(\zeta_{1i},r')\cos\frac{n\pi z'}{L}dr'dz'\right]\frac{\zeta_{1i}^2J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}K_1(\zeta_{1i},r)\right\} \qquad (29)$$

By interchanging integrals with summations, we may write Eq. (29) in the form

$$\Psi_r(r,z)=\int_{-L}^{L}\int_{a}^{b}X_r(r',z')G_r(r,r',z,z')dr'dz' \qquad (30)$$

where

$$G_r(r,r',z,z')=-\sum_{i=1}^{\infty}\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}r'\times K_1(\zeta_{1i},r)K_1(\zeta_{1i},r')\times\left[\frac{1}{2L\zeta_{1i}^4}+L^3\sum_{n=1}^{\infty}\frac{1}{(L^2\zeta_{1i}^2+n^2\pi^2)^2}\times\cos\frac{n\pi z'}{L}\cos\frac{n\pi z}{L}\right]\right\} \qquad (31)$$

is the radial Green's function.

5 Axial Green's Function

The axial Green's function is obtained by solving the partial differential equation for the axial stress function, Eq. (9), with Dirichlet homogeneous boundary conditions. Here, finite Hankel transform in the radial direction and finite Fourier sine transform in the axial direction (since the axial body force is an odd function of z) are used.

Applying the finite Hankel transform of order zero in r, and finite Fourier since transform in z, to Eq. (9) yields

$$\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)^2\bar{\bar{\Psi}}_z(\zeta_{0i},n)=-\bar{\bar{X}}_z(\zeta_{0i},n) \qquad (32)$$

where n is an integer, ζ_{0i} satisfies

$$J_0(\zeta_{0i}a)Y_0(\zeta_{0i}b)-J_0(\zeta_{0i}b)Y_0(\zeta_{0i}a)=0 \qquad (33)$$

and

$$\bar{\bar{\Psi}}_z(\zeta_{0i},n)=\Re_0[\Im_s[\Psi_z(r,z)]]=\int_{-L}^{L}\int_{a}^{b}r\Psi_z(r,z)K_0(\zeta_{0i},r)\sin\frac{n\pi z}{L}drdz \qquad (34)$$

$$\bar{\bar{X}}_z(\zeta_{0i},n)=\Re_0[\Im_s[X_z(r,z)]]=\int_{-L}^{L}\int_{a}^{b}rX_z(r,z)K_0(\zeta_{0i},r)\sin\frac{n\pi z}{L}drdz \qquad (35)$$

are transforms of the axial stress function and axial body force. Here,

$$K_0(\zeta_{0i},r)=[J_0(\zeta_{0i}r)Y_0(\zeta_{0i}b)-J_0(\zeta_{0i}b)Y_0(\zeta_{0i}r)] \qquad (36)$$

is the Fourier Bessel kernel for the zero-order transformation. The inverse transform of $\bar{\bar{\Psi}}_z(\zeta_{0i},n)$ is

$$\Psi_z(r,z)=\Re_0^{-1}[\Im_s^{-1}[\bar{\bar{\Psi}}_z(\zeta_{0i},n)]]=\frac{\pi^2}{2L}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{\zeta_{0i}^2J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\times\sin\frac{n\pi z}{L}\bar{\bar{\Psi}}_z(\zeta_{0i},n)K_0(\zeta_{0i},r). \qquad (37)$$

Incorporating Eqs. (32) and (35) into Eq. (37) gives the solution to the axial stress function $\Psi_z(r,z)$ in terms of the axial body force $X_z(r,z)$.

$$\Psi_z(r,z)=\int_{-L}^{L}\int_{a}^{b}X_z(r',z')G_z(r,r',z,z')dr'dz' \qquad (38)$$

where

$$G_z(r,r',z,z')=-\frac{\pi^2}{2}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{L^3}{(L^2\zeta_{0i}^2+n^2\pi^2)^2}\frac{\zeta_{0i}^2J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}r'K_0(\zeta_{0i},r')K_0(\zeta_{0i},r)\sin\frac{n\pi z'}{L}\sin\frac{n\pi z}{L}\right] \qquad (39)$$

is the axial Green's function.

6 Boundary Conditions

The displacement vector is related to the vector $\boldsymbol{\Psi}$ by Eq. (40).

$$\mathbf{u}=-\frac{1}{\mu}\nabla\times(\nabla\times\boldsymbol{\Psi})+\frac{1}{\lambda+2\mu}\nabla(\nabla.\boldsymbol{\Psi}) \tag{40}$$

The stress tensor in terms of the displacement vector is defined by

$$\boldsymbol{\sigma}=\lambda(\nabla.\mathbf{u})\mathbf{I}+\mu[\nabla\mathbf{u}+\nabla^T\mathbf{u}] \tag{41}$$

where $\mathbf{I}$ is the identity tensor. Substituting Eq. (40) into Eq. (41) results in stresses in terms of axial and radial stress functions

$$\sigma_r=2\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+\frac{1}{1-\nu}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\varphi$$

$$\sigma_\theta=\frac{2}{r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+\frac{1}{1-\nu}\left(\nu\nabla^2-\frac{1}{r}\frac{\partial}{\partial r}\right)\varphi$$

$$\sigma_z=2\frac{\partial}{\partial z}(\nabla^2\Psi_z)+\frac{1}{1-\nu}\left(\nu\nabla^2-\frac{\partial^2}{\partial z^2}\right)\varphi$$

$$\sigma_{rz}=\frac{\partial}{\partial r}(\nabla^2\Psi_z)+\frac{\partial}{\partial z}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r-\frac{1}{1-\nu}\frac{\partial^2\varphi}{\partial r\partial z} \tag{42}$$

where ν is the Poisson's ratio and

$$\varphi=\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)+\frac{\partial\Psi_z}{\partial z} \tag{43}$$

is the divergence of the vector $\boldsymbol{\Psi}$.

Traction-free boundary conditions are appropriate for a solenoid coil. Therefore, the radial and shear stresses should be zero at the inside and outside radii ($r=a$ and $r=b$), and the axial and shear stresses should be zero at the ends of the coil ($z=\pm L$).

Substituting the solutions for radial and axial stress functions from Eqs. (30), (31), (38), and (39) into Eq. (42) and computing radial and shear stresses at the inside and outside radii, and the axial and shear stresses at the ends of the coil, yields

$$\sigma_r(a,z)=\sum_{n=0}^{\infty}\wp_1(n)\cos\frac{n\pi z}{L} \tag{44}$$

$$\sigma_{rz}(a,z)=\sum_{n=1}^{\infty}\wp_2(n)\sin\frac{n\pi z}{L}$$

$$\sigma_r(b,z)=\sum_{n=0}^{\infty}\wp_3(n)\cos\frac{n\pi z}{L}$$

$$\sigma_{rz}(b,z)=\sum_{n=1}^{\infty}\wp_4(n)\sin\frac{n\pi z}{L}$$

$$\sigma_z(r,\pm L)=\sum_{i=1}^{\infty}[\wp_5(\zeta_{0i})K_0(\zeta_{0i},r)+\wp_6(\zeta_{1i})K_0^*(\zeta_{1i},r)]$$

$$\sigma_{rz}(r,\pm L)=0$$

where

$$K_0^*(\zeta_{1i},r)=[J_0(\zeta_{1i}r)Y_1(\zeta_{1i}b)-J_1(\zeta_{1i}b)Y_0(\zeta_{1i}r)]$$
$$=\frac{1}{r\zeta_{1i}}K_1(\zeta_{1i},r)+\frac{1}{\zeta_{1i}}\frac{\partial}{\partial r}K_1(\zeta_{1i},r), \tag{45}$$

and $\wp_1(n)$ through $\wp_4(n)$, $\wp_5(\zeta_{0i})$ and $\wp_6(\zeta_{1i})$ are given by

$$\wp_1(n)=\sum_{i=1}^{\infty}\frac{2}{\pi a^2}\frac{1}{1-\nu}\left\{-\Gamma_z(\zeta_{0i},n)\frac{n\pi}{L}\frac{J_0(\zeta_{0i}b)}{J_0(\zeta_{0i}a)}+a\Gamma_r(\zeta_{1i},n)\times\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\frac{J_1(\zeta_{0i}b)}{J_1(\zeta_{0i}a)}\right\} \tag{46}$$

$$\wp_2(n)=\sum_{i=1}^{\infty}\frac{2}{\pi a}\frac{1}{1-\nu}\left\{-\Gamma_z(\zeta_{0i},n)\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]\frac{J_0(\zeta_{0i}b)}{J_0(\zeta_{0i}a)}\right\}$$

$$\wp_3(n)=\sum_{i=1}^{\infty}\frac{2}{\pi b^2}\frac{1}{1-\nu}\left\{-\Gamma_z(\zeta_{0i},n)\frac{n\pi}{L}+b\Gamma_r(\zeta_{1i},n)\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\right\}$$

$$\wp_4(n)=\sum_{i=1}^{\infty}\frac{1}{1-\nu}\frac{2}{\pi b}\left\{-\Gamma_z(\zeta_{0i},n)\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]\right\}$$

$$\wp_5(\zeta_{0i})=\sum_{n=0}^{\infty}\frac{(-1)^n}{1-\nu}\left\{-\Gamma_z(\zeta_{0i},n)\frac{n\pi}{L}\left[\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{0i}^2(2-\nu)\right]\right\}$$

$$\wp_6(\zeta_{1i})=\sum_{n=0}^{\infty}\frac{(-1)^n}{1-\nu}\left\{\Gamma_r(\zeta_{1i},n)\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]\right\}$$

with

$$\Gamma_z(\zeta_{0i},n)=-\frac{\pi^2}{2}\left\{\frac{L^3}{(L^2\zeta_{0i}^2+n^2\pi^2)^2}\frac{\zeta_{0i}^2J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\times\int_{-L}^{L}\int_a^b r'X_z(r',z')K_0(\zeta_{0i},r')\sin\frac{n\pi z'}{L}dr'dz'\right\} \tag{47}$$

$$\Gamma_r(\zeta_{1i},n)=-\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}\frac{L^3}{(L^2\zeta_{1i}^2+n^2\pi^2)^2}\times\int_{-L}^{L}\int_a^b r'X_r(r',z')K_1(\zeta_{1i},r')\cos\frac{n\pi z'}{L}dr'dz'\right\}$$

$$\Gamma_r(\zeta_{1i},0)=-\frac{\pi^2}{2}\frac{J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a)-J_1^2(\zeta_{1i}b)}\frac{1}{2L\zeta_{1i}^2}\times\int_{-L}^{L}\int_a^b r'X_r(r',z')K_1(\zeta_{1i},r')dr'dz'.$$

From Eq. (44), it can be observed that except for the shear stress at the ends of the coil, boundary conditions are not satisfied. The radial and shear stresses impose forcing functions of z at the radial boundaries and the axial stress asserts a forcing function of r at the axial boundaries. Thus, a complementary solution for either radial or axial stress function (since stresses are related to both) is needed to neutralize these forcing functions.

7 Complementary Solution for the Axial Stress Function

Let us consider function $\xi(r,z)$ (an odd function in z) as a complementary function for the axial stress function. From Eq. (9), $\xi(r,z)$ must satisfy the homogeneous part of the partial differential equation for the axial stress function.

$$\nabla^4\zeta(r,z)=0 \tag{48}$$

From Eq. (42) radial, axial, and shear stresses are expressed in terms of $\xi(r,z)$.

$$\sigma_r=\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\xi(r,z) \tag{49}$$

$$\sigma_z=\frac{1}{1-\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi(r,z)$$

$$\sigma_{rz}=\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi(r,z)$$

The complementary function must reverse the effect of the imposed forcing functions by stresses at the boundaries. As a result, from Eqs. (44) and (49) boundary conditions for $\xi(r,z)$ are obtained and given by

$$\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\xi(r,z)\bigg|_{r=a}=-\sum_{n=0}^{\infty}\wp_1(n)\cos\frac{n\pi z}{L} \tag{50}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi(r,z)\bigg|_{r=a}=-\sum_{n=1}^{\infty}\wp_2(n)\sin\frac{n\pi z}{L}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\xi(r,z)\bigg|_{r=b}=-\sum_{n=0}^{\infty}\wp_3(n)\cos\frac{n\pi z}{L}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi(r,z)\bigg|_{r=b}=-\sum_{n=1}^{\infty}\wp_4(n)\sin\frac{n\pi z}{L}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi(r,z)\bigg|_{z=L}$$
$$=-\sum_{i=1}^{\infty}\left[\wp_5(\zeta_{0i})K_0(\zeta_{0i},r)+\wp_6(\zeta_{1i})K_0^*(\zeta_{1i},r)\right]$$
$$\times\frac{1}{1-\nu}\frac{\partial}{\partial r}\left((1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right)\xi(r,z)\bigg|_{z=L}=0.$$

The partial differential equation for $\xi(r,z)$ with the given boundary conditions may be solved by using the superposition principle. The substitution of $\xi(r,z)=\xi_1(r,z)+\xi_2(r,z)$ into Eq. (48) yields two partial differential equations for $\xi_1(r,z)$ and $\xi_2(r,z)$.

$$\nabla^4\xi_1(r,z)=0 \tag{51}$$

$$\nabla^4\xi_2(r,z)=0 \tag{52}$$

Solution to $\xi_1(r,z)$ is achieved by applying the finite Hankel transform of order zero to Eq. (51) and solving the resulting differential equation for z.

$$\xi_1(r,z)=\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 I_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}[A_i\sinh(\zeta_{0i}z)+B_i z\cosh(\zeta_{0i}z)]K_0(\zeta_{0i},r) \tag{53}$$

Here, A_i and B_i are arbitrary constants. Solution to $\xi_2(r,z)$ is obtained by employing the finite Fourier sine transform to Eq. (52) and solving the ensuing differential equation for r.

$$\xi_2(r,z)=\frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\hat{A}_n r I_1\left(\frac{n\pi}{L}r\right)+\frac{1}{n}\hat{B}_n r K_1\left(\frac{n\pi}{L}r\right)+\frac{2\pi}{L}\hat{C}_n I_0\left(\frac{n\pi}{L}r\right)+\frac{2\pi}{L}\hat{D}_n K_0\left(\frac{n\pi}{L}r\right)\right]\sin\frac{n\pi z}{L} \tag{54}$$

In Eq. (54), $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, and $\hat{D}_n$ are arbitrary constants, $I_0[(n\pi/L)r]$ and $I_1[(n\pi/L)r]$ are the modified Bessel functions of the first kind, and $K_0[(n\pi/L)r]$ and $K_1[(n\pi/L)r]$ are the modified Bessel functions of the second kind. The superposition of Eqs. (53) and (54) furnishes the solution to $\xi(r,z)$.

$$\xi(r,z)=\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}[A_i\sinh(\zeta_{0i}z)+B_i z\cosh(\zeta_{0i}z)]K_0(\zeta_{0i},r)+\frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\hat{A}_n r I_1\left(\frac{n\pi}{L}r\right)+\frac{1}{n}\hat{B}_n r K_1\left(\frac{n\pi}{L}r\right)+\frac{2\pi}{L}\hat{C}_n I_0\left(\frac{n\pi}{L}r\right)+\frac{2\pi}{L}\hat{D}_n K_0\left(\frac{n\pi}{L}r\right)\right]\sin\frac{n\pi z}{L} \tag{55}$$

The six arbitrary constants in Eq. (55) may be evaluated by using the six boundary conditions given by Eq. (50). Applying the shear stress boundary condition at $z=L$ yields

$$B_i=\omega(\zeta_{0i})A_i \tag{56}$$

where $\omega(\zeta_{0i})$ is expressed by Eq. (57).

$$\omega_{(\zeta_{0i})}=\frac{-\zeta_{0i}}{2\nu+\zeta_{0i}L\coth(\zeta_{0i}L)} \tag{57}$$

Employing the radial boundary conditions to $\xi(r,z)$ and using Eq. (57) results in

$$\varkappa_{11}(n)\hat{A}_n+\varkappa_{12}(n)\hat{B}_n+\varkappa_{13}(n)\hat{C}_n+\varkappa_{14}(n)\hat{D}_n+\sum_{i=1}^{\infty}\Lambda_1(\zeta_{0i},n)A_i=\wp_1(n) \tag{58}$$

$$\varkappa_{21}(n)\hat{A}_n+\varkappa_{22}(n)\hat{B}_n+\varkappa_{23}(n)\hat{C}_n+\varkappa_{24}(n)\hat{D}_n+\sum_{i=1}^{\infty}\Lambda_2(\zeta_{0i},n)A_i=\wp_2(n) \tag{59}$$

$$\varkappa_{31}(n)\hat{A}_n+\varkappa_{32}(n)\hat{B}_n+\varkappa_{33}(n)\hat{C}_n+\varkappa_{34}(n)\hat{D}_n+\sum_{i=1}^{\infty}\Lambda_3(\zeta_{0i},n)A_i=\wp_3(n) \tag{60}$$

$$\varkappa_{41}(n)\hat{A}_n+\varkappa_{42}(n)\hat{B}_n+\varkappa_{43}(n)\hat{C}_n+\varkappa_{44}(n)\hat{D}_n+\sum_{i=1}^{\infty}\Lambda_4(\zeta_{0i},n)A_i=\wp_4(n) \tag{61}$$

where $\varkappa_{11}(n)$ through $\varkappa_{44}(n)$ and $\Lambda_1(\zeta_{0i},n)$ through $\Lambda_4(\zeta_{0i},n)$ are given in Appendix A. The boundary condition for axial stress at $z=L$ provides

$$\Lambda_5(\zeta_{0i})A_i+\sum_{n=1}^{\infty}[\varkappa_{51}(\zeta_{0i},n)\hat{A}_n+\varkappa_{52}(\zeta_{0i},n)\hat{B}_n+\varkappa_{53}(\zeta_{0i},n)\hat{C}_n+\varkappa_{54}(\zeta_{0i},n)\hat{D}_n]=\Gamma_5(\zeta_{0i},\zeta_{1i}) \tag{62}$$

where $\varkappa_{51}(\zeta_{0i},n)$ through $\varkappa_{54}(\zeta_{0i},n)$, $\Lambda_5(\zeta_{0i})$ and $\Gamma_5(\zeta_{0i},\zeta_{1i})$ are given in Appendix B.

Equations (58)–(62) represent a system of equations where the unknowns are $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, $\hat{D}_n$ and A_i. To evaluate these unknowns, the infinite series in Eqs. (58)–(62) are replaced by finite summations with an acceptable truncation error. Hence, the infinite upper limits for i and n are changed to finite values of M and N, respectively. Expanding these finite summations would result in a system of equations with $4N+M$ unknowns and equations, where unknowns are $\hat{A}_1-\hat{A}_N$, $\hat{B}_1-\hat{B}_N$, $\hat{C}_1-\hat{C}_N$, $\hat{D}_1-\hat{D}_N$ and A_1-A_M. Equation (62) gives M equations by letting i vary from 1 to M. Moreover, allowing n to advance from 1 to N in Eqs. (58)–(61), results in $4N$ equations. By solving this system of equations, the arbitrary constants for the complementary solution of the axial stress function are obtained. The combination of the

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complementary and the Green's function solutions for the axial stress function yields a solution that satisfies both the boundary conditions and the axial body force. This solution together with the radial Green's function determines the distribution of stresses in a given coil.

8 Numerical Results

The Green's function solution is applied to a 23 Tesla superconducting coil. The parameters for this coil are given in Table 1. Figures 2 and 3 show the tangential and radial stresses through the coil along the radius at three different axial positions ($z=0$, $z=L/2$ and $z=L$). Figure 4 shows the characteristics of the axial stress through the coil along the radius at the midplane and $z=L/2$; and Fig. 5 shows the shear stress at $z=L/2$. Note that due to traction free boundary conditions, axial and shear stresses are zero at $z=L$ and shear stress is zero at the midplane ($z=0$) because of symmetry.

Table 1 Parameters for the 23 T superconducting coil

Name	Symbol	Value	Unit
Inner radius	a	100.00	mm
Outer radius	b	136.50	mm
Half length	L	28.00	mm
Elastic modulus	E	111.00	GPa
Poisson's ratio	ν	0.30	
Current density	J	530.10	A/mm^2

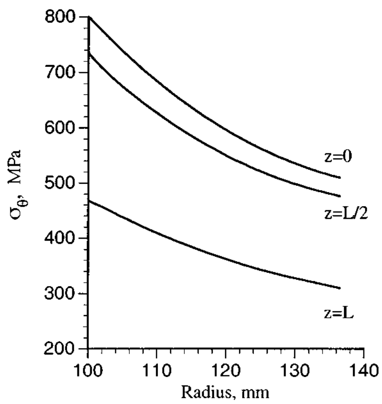

Fig. 2 Distribution of the tangential stress for a 23 Tesla superconducting coil

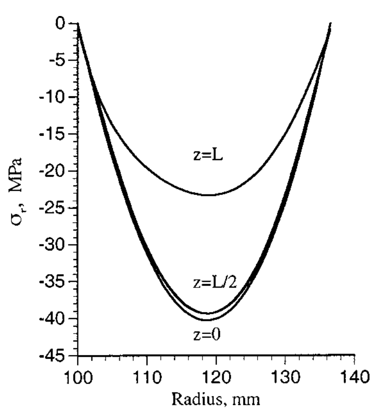

Fig. 3 Distribution of the radial stress for a 23 Tesla superconducting coil

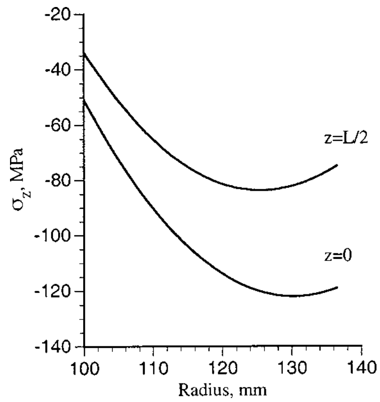

Fig. 4 Distribution of the axial stress for a 23 Tesla superconducting coil

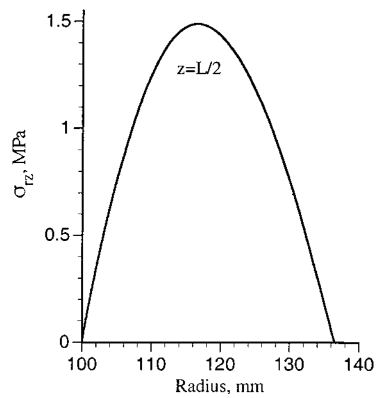

Fig. 5 Distribution of the shear stress for a 23 Tesla superconducting coil

9 Conclusions

Analytical closed-form solution for the distribution of stresses has been developed for a coil of high-field solenoid magnets, including superconducting magnets. This solution is presented in forms of the Green's functions, which permits the development of a solution irrespective of the type of the field or its distribution within a coil. The problem was formulated in terms of stress functions. Green's functions were derived by using finite Hankel and finite Fourier transforms. Boundary conditions were satisfied by introducing a complementary solution for the axial stress function. The radial Green's function with the superposition of the complementary and the axial Green's function provide a comprehensive analytical solution for the stresses.

The Green's function solution provides a complete analytical stress solution for an isotropic coil. This solution should be used as a foundation for the stress analysis of multilayer magnets. The future work should also extend this solution for an orthotropic coil.

Appendix A

$$\chi_{11}(n)=\frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}I_0\left(\frac{n\pi}{L}a\right)-\frac{n^2\pi^2}{2L^3}aI_1\left(\frac{n\pi}{L}a\right)\right]$$

$$\chi_{12}(n)=\frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}K_0\left(\frac{n\pi}{L}a\right)-\frac{n^2\pi^2}{2L^3}aK_1\left(\frac{n\pi}{L}a\right)\right]$$

$$\chi_{13}(n)=\frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_0\left(\frac{n\pi}{L}a\right)+\frac{n^2\pi^2}{L^3}\frac{1}{a}I_1\left(\frac{n\pi}{L}a\right)\right]$$

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$$\varkappa_{14}(n)=\frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_0\left(\frac{n\pi}{L}a\right)+\frac{n^2\pi^2}{L^3}\frac{1}{a}K_1\left(\frac{n\pi}{L}a\right)\right]$$

$$\varkappa_{21}(n)=\frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}I_0\left(\frac{n\pi}{L}b\right)-\frac{n^2\pi^2}{2L^3}bI_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{22}(n)=\frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}K_0\left(\frac{n\pi}{L}b\right)-\frac{n^2\pi^2}{2L^3}bK_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{23}(n)=\frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_0\left(\frac{n\pi}{L}b\right)+\frac{n^2\pi^2}{L^3}\frac{1}{b}I_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{24}(n)=\frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_0\left(\frac{n\pi}{L}b\right)+\frac{n^2\pi^2}{L^3}\frac{1}{b}K_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{31}(n)=-\left[\frac{n\pi}{L^2}I_1\left(\frac{n\pi}{L}a\right)+\frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aI_0\left(\frac{n\pi}{L}a\right)\right]$$

$$\varkappa_{32}(n)=-\left[\frac{n\pi}{L^2}K_1\left(\frac{n\pi}{L}a\right)+\frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aK_0\left(\frac{n\pi}{L}a\right)\right]$$

$$\varkappa_{33}(n)=-\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_1\left(\frac{n\pi}{L}a\right)\right]$$

$$\varkappa_{34}(n)=-\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_1\left(\frac{n\pi}{L}a\right)\right]$$

$$\varkappa_{41}(n)=-\left[\frac{n\pi}{L^2}I_1\left(\frac{n\pi}{L}b\right)+\frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bI_0\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{41}(n)=-\left[\frac{n\pi}{L^2}I_1\left(\frac{n\pi}{L}b\right)+\frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bI_0\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{42}(n)=-\left[\frac{n\pi}{L^2}K_1\left(\frac{n\pi}{L}b\right)+\frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bK_0\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{43}(n)=-\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\varkappa_{44}(n)=-\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_1\left(\frac{n\pi}{L}b\right)\right]$$

$$\Lambda_1(\zeta_{0i},n)=\frac{1}{(1-\nu)}\frac{\pi}{a^2}\frac{2\zeta_{0i}^2J_0(\zeta_{0i}a)J_0(\zeta_{0i}b)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\{\zeta_{0i}\Omega_2(\zeta_{0i},n)+\omega(\zeta_{0i})[\zeta_{0i}\Omega_1(\zeta_{0i},n)+\Omega_2(\zeta_{0i},n)]\}$$

$$\Lambda_2(\zeta_{0i},n)=\frac{1}{(1-\nu)}\frac{\pi}{b^2}\frac{2\zeta_{0i}^2J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\{\zeta_{0i}\Omega_2(\zeta_{0i},n)+\omega(\zeta_{0i})[\zeta_{0i}\Omega_1(\zeta_{0i},n)+\Omega_2(\zeta_{0i},n)]\}$$

$$\Lambda_3(\zeta_{0i},n)=\frac{1}{(1-\nu)}\frac{\pi}{a}\frac{2\zeta_{0i}^3J_0(\zeta_{0i}a)J_0(\zeta_{0i}b)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\{-\zeta_{0i}\Omega_4(\zeta_{0i},n)+\omega(\zeta_{0i})[2\nu\Omega_4(\zeta_{0i},n)-\zeta_{0i}\Omega_3(\zeta_{0i},n)]\}$$

$$\Lambda_4(\zeta_{0i},n)=\frac{1}{(1-\nu)}\frac{\pi}{b}\frac{2\zeta_{0i}^3J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\{-\zeta_{0i}\Omega_4(\zeta_{0i},n)+\omega(\zeta_{0i})[2\nu\Omega_4(\zeta_{0i},n)-\zeta_{0i}\Omega_3(\zeta_{0i},n)]\}$$

where

$$\Omega_1(\zeta_{0i},n)=\int_{-L}^{L}z\cos\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz$$
$$=\frac{2L^2(-1)^n}{(\zeta_{0i}^2L^2+n^2\pi^2)^2}[-(\zeta_{0i}^2L^2-n^2\pi^2)\sinh(\zeta_{0i}L)+(\zeta_{0i}^2L^2+n^2\pi^2)\zeta_{0i}L\cosh(\zeta_{0i}L)]$$

$$\Omega_2(\zeta_{0i},n)=\int_{-L}^{L}\cos\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz$$
$$=\frac{2\zeta_{0i}L^2(-1)^n}{(\zeta_{0i}^2L^2+n^2\pi^2)}\sinh(\zeta_{0i}L)$$

$$\Omega_3(\zeta_{0i},n)=\int_{-L}^{L}z\sin\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz$$
$$=\frac{2L^2(-1)^n}{(\zeta_{0i}^2L^2+n^2\pi^2)^2}[2n\pi L\zeta_{0i}\sinh(\zeta_{0i}L)-(\zeta_{0i}^2L^2+n^2\pi^2)n\pi\cosh(\zeta_{0i},L)]$$

$$\Omega_4(\zeta_{0i},n)=\int_{-L}^{L}\sin\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz$$
$$=\frac{-2n\pi L(-1)^n}{(\zeta_{0i}^2L^2+n^2\pi^2)}\sinh(\zeta_{0i}L).$$

Appendix B

$$\varkappa_{51}(\zeta_{0i},n)=(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)I_0\left(\frac{n\pi}{L}r\right)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)I_1\left(\frac{n\pi}{L}r\right)dr\right]$$

$$\varkappa_{52}(\zeta_{0i},n)=(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)K_0\left(\frac{n\pi}{L}r\right)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)K_1\left(\frac{n\pi}{L}r\right)dr\right]$$

$$\varkappa_{53}(\zeta_{0i},n)=(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)I_0\left(\frac{n\pi}{L}r\right)dr\right]$$

$$\varkappa_{54}(\zeta_{0i},n)=(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)K_0\left(\frac{n\pi}{L}r\right)dr\right]$$

$$\Lambda_5(\zeta_{0i})=2\zeta_{0i}^2[(1-2\nu)\omega(\zeta_{0i})\cosh(\zeta_{0i}L)-\zeta_{0i}\cosh(\zeta_{0i}L)-\zeta_{0i}\omega(\zeta_{0i})L\sinh(\zeta_{0i}L)]$$

$$\Gamma_5(\zeta_{0i},\zeta_{1i})=-(1-\nu)\left\{\frac{2}{\pi^2\zeta_{0i}^2}\left[1-\left(\frac{J_0(\zeta_{0i}b)}{J_0(\zeta_{0i}a)}\right)^2\right]\wp_5(\zeta_{0i})+\wp_6(\zeta_{1i})\int_a^b rK_0(\zeta_{0i},r)K_0^*(\zeta_{1i},r)dr\right\}$$

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